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THE SOLUTION OF RICCATI'S EQUATION AS THE HESSIAN OF BELLMAN'S FUNCTION†

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The problem of optimal control with separated conditions for the ends is investigated. It is assumed that for the manifold of left ends (and also for the manifold of right ends) a field of extremals including the given extremal exists. A criterion, which gives the necessary and sufficient conditions for optimality in terms of these two fields, is proved. The positive definiteness of the difference of the solutions of the corresponding Riccati matrix equations serves as the sufficient condition, and its non-negativity serves as the necessary condition. The formula relating the solution of Riccati's equation to the Hessian of Bellman's function plays a key role in the proof of the criterion.

1. FORMULATION OF THE PROBLEM

Consider the problem of minimizing the functional

$$J(u(\cdot)) = \int_{t_1}^{t_2} f(t, x, u)dt \tag{1.1}$$

with the constraints

$$\dot{x} = \varphi(t, x, u), \quad \Phi_1(t_1, x(t_1)) = 0, \quad \Phi_2(t_2, x(t_2)) = 0; \quad u(t) \in U \tag{1.2}$$

Here x are the phase variables, belonging to a continuous n-dimensional manifold M, the control $u(t) \in U$ is continuous and the functions f, φ, Φ_1, Φ_2 depend continuously on their arguments.

Note that the results obtained in this paper hold for considerably less rigid assumptions, but for clarity and simplicity, we will derive the simplest version here. Without loss of generality we will assume that $f(t, x, u) > 0$. This can always be supplemented by adding an appropriate constant to the integrand. We will denote the subset of left ends (the second equality of (1.2)) by $M_1 \subset \mathbb{R} \times M$ and the subset of the right ends (the third equality of (1.2)) by $M_2 \subset \mathbb{R} \times M$; the dimensions of M_1 and M_2 are arbitrary.

2. PRELIMINARY FACTS

We will recall some facts which touch on problems (1.1) and (1.2). Suppose ψ is an element of a cotangent bundle T^*M of the manifold M. We will conditionally denote the result of substituting the functions $\hat{x}(t)$, $\hat{\psi}(t)$, $\hat{u}(t)$ under the sign of a certain function $F(x, \psi, u)$ by $\hat{F}(t)$. Consider the Pontryagin function

$$\mathcal{H}(t, x, \psi, u) = -f(t, x, u) + \psi\varphi(t, x, u)$$

We will denote its maximum with respect to u by $H(t, x, \psi)$. We will assume that the problem is normal, and hence the coefficient of f can be taken to be equal to -1. Everywhere henceforth we will also assume

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that the value of u, which gives a maximum of $\mathcal{H}$, is uniquely defined and the function $H(t, x, \psi)$ is continuous.

Suppose $\hat{x}(t)$, $\hat{u}(t)$, $t \in [\hat{t}_1, \hat{t}_2]$ gives a strong local minimum. Then, by virtue of Pontryagin's maximum principle, a continuous lifting $\hat{\psi}(t)$ of the optimum trajectory $x(t)$ exists in the co-tangent bundle, which satisfies the following conditions.

1. The function $\mathcal{H}(t, \hat{x}(t), \hat{\psi}(t), u)$ reaches its maximum with respect to u when $u = \hat{u}(t)$

$$H(t, \hat{x}(t), \hat{\psi}(t)) = \hat{H}(t) = \max_{u \in U} \mathcal{H}(t, \hat{x}(t), \hat{\psi}(t), u) \tag{2.1}$$

2. The pair of functions $\hat{x}(\cdot)$, $\hat{\psi}(\cdot)$ is the solution of the Hamilton system

$$\dot{x} = H_{\psi}(t, x, \psi), \quad \dot{\psi} = -H_x(t, x, \psi) \tag{2.2}$$

3. The conditions of transversality are satisfied: the pair $(-\hat{H}(\hat{t}_k), \hat{\psi}(\hat{t}_k))$ is an annihilator of the tangential plane to the submanifold M_k at the point $\hat{t}_k$, $\hat{x}(\hat{t}_k)$, i.e.

$$-\hat{H}(\hat{t}_k)\theta + \hat{\psi}(\hat{t}_k)\xi = 0 \quad \text{for all} \quad (\theta, \xi) \in T_* M_k(\hat{t}_k, \hat{x}(\hat{t}_k)) \tag{2.3}$$

Here $k = 1$ for the left end and $k = 2$ for the right end.

Assumption 1. The function $\psi(\cdot)$, which satisfies conditions (2.1)–(2.3), is uniquely defined.

The pairs $x(\cdot)$ and $u(\cdot)$, which satisfy conditions (2.1)–(2.3), are called extremals.

We will define lifting $\mathfrak{M}_1$ of the manifold M_1 in extended phase space of the variables t, x, ψ: for each point $(t, x) \in M_1$ we consider all ψ which satisfy condition (2.3) at this point. It is easy to see that $\dim \mathfrak{M}_1 = n$. We will denote the lifting of the manifold M_2 by $\mathfrak{M}_2$.

Assumption 2. We will assume that the velocity vector of the Hamilton system (2.2), supplemented by the equation $\dot{t} = 1$:

$$\zeta_k = (1, \hat{H}_{\psi}(\hat{t}_k), -\hat{H}_x(\hat{t}_k)) \tag{2.4}$$

does not touch the manifold $\mathfrak{M}_k$, $(k = 1, 2)$.

Assumption 2 is preserved for a certain neighbourhood $\mathcal{U}_k \subset \mathfrak{M}_k$ of the point $(\hat{t}_k, \hat{x}(\hat{t}_k), \hat{\psi}(\hat{t}_k))$.

We will denote the n-dimensional vector of the coordinates, which parametrize $\mathcal{U}_k$, by σ_k.

Consider the solutions of Hamilton system (2.2) with initial conditions at the points $\sigma_1 = (t_1, x(t_1), \psi(t_1)) \in \mathcal{U}_1$. Propositions 1 and 2 guarantee that, as a result, an $(n + 1)$-dimensional continuous manifold $\mathfrak{N}_1$ is obtained, which, by virtue of the Poincaré-Cartan integral invariant theorem and by virtue of transversality conditions on the left end (2.3), is Lagrangian, i.e.

$$\oint_{\gamma} (-Hdt + \psi dx) = 0 \tag{2.5}$$

for any closed curve $\gamma \subset \mathfrak{N}_1$. By decreasing the neighbourhood $\mathcal{U}_1$, if necessary, we require that the Manifold $\mathfrak{N}_1$ when $t \in (t_1, t_2]$, where t_1 and t_2 depend on the corresponding trajectory $(x(t), \psi(t)) \subset \mathfrak{N}_1$, is projected diffeomorphically onto a certain region N_1 of the space (t, x). If this is possible we say that a field of extremals $\mathfrak{P}_1$ corresponding to the manifold M_1 is defined in N_1, and that there are no focal points of the manifold M_1 on this extremal. In this case the projections of the extremals which lie in $\mathfrak{N}_1$, uniquely cover the region N_1. In view of the one-to-one nature of the projection, the function $\psi_1(t, x)$ is defined in N_1 and then $-Hdt + \psi dx$ becomes a differential form on N_1, and equality (2.5) denotes that this form is exact. Consequently, a function $S_1(t, x)$ exists such that

$$dS_1 = -Hdt + \psi dx \tag{2.6}$$

Hence it follows that the Hamilton–Jacobi equation in the Bellman form is satisfied for S_1, while the function S_1 is Bellman's function in the problem of minimizing the functional J with initial manifold M_1 and with the right end at the point (t, x).

We carry out exactly the same construction, but only with motion in time backwards, for manifold M_2, and we obtain another solution $S_2(t, x)$ of the Hamilton–Jacobi equation, corresponding to the manifold of the ends M_2.

Assumption 3. For the extremal $x(\cdot)$, $u(\cdot)$ there are no focal points of the manifold M_1 in the half-interval $(\hat{t}_1, \hat{t}_2]$. These are no focal points of the manifold M_2 in the half-interval $[\hat{t}_1, \hat{t}_2)$.

If, say, the manifold M_2 was reduced to a point, the lack of focal points of the manifold M_1 together with the Pontryagin maximum principle would give a sufficient condition for a strong minimum, since the extremal $x(\cdot)$ would be imbedded in the field, and the condition for a maximum of Pontryagin's function would guarantee that Weiserstrass's function is non-negative. However, if both manifolds, M_1 and M_2, are non-trivial, Assumption 3 is necessary, but far from being sufficient for optimality.

This paper is devoted to finding an important formula for calculating the Hessian (the matrix of the second differential) of the functions $S_k(t, x)$. This formula is not simply a new interpretation of existing constructions, but a convenient mathematical apparatus. In particular, it has enabled us to obtain the necessary and sufficient conditions for problem (1.1), (1.2) to be optimal, derived below, that are the simplest and most effective as a criterion for there to be no focal point for a problem with one clamped end.

3. FUNDAMENTAL THEOREMS

Consider the following system of equations in variations for Eqs (2.2)

$$\begin{aligned} \dot{q} &= H_{\psi x}(t, x, \psi)q + H_{\psi\psi}(t, x, \psi)p \\ \dot{p} &= -H_{xx}(t, x, \psi)q - H_{x\psi}(t, x, \psi)p \end{aligned} \tag{3.1}$$

We have denoted by q and p the derivatives with respect to the initial data for the functions x and ψ respectively, which are solutions of system (2.2). Henceforth, it will be more convenient to regard q and p as being $n \times n$ matrices of derivatives with respect to the quantities σ_1. The coefficients of system (3.1) are also $n \times n$ matrices.

Riccati's matrix equation [1] for the transfer of Lagrange planes with respect to the solutions of system (3.1) will be the main instrument of the investigation. We will denote the matrix coordinates of the Lagrange planes by $W = pq^{-1}$. One can easily obtain, by direct differentiation, the following Riccati matrix equation of W

$$-\dot{W} = H_{xx} + H_{x\psi}W + WH_{\psi x} + WH_{\psi\psi}W \tag{3.2}$$

Since the matrix of the coefficients of system (3.1) belong to a Lie algebra of a symplectic Lie group, the matrix $W(t)$ will be symmetric if its initial value $W(t_1)$ is a symmetric matrix. The initial value $W(t_1)$ is found from the transversality condition (2.3), which gives the Lagrange plane and, consequently, is a symmetric matrix.

If the matrix q is degenerate, the solution $W(t)$ of Eq. (3.2) departs to infinity (a focal point), and if it is necessary to start or extend the solution of Eq. (3.2) one must perform a matrix fractional-linear transformation in order to transfer to another map in a Lagrange-Grassman manifold.

Theorem 1. We will assume that Assumptions 1, 2 and 3 are satisfied on the extremal $\hat{x}(\cdot)$, $\hat{\psi}(\cdot)$. Suppose the solution $p(t)$, $q(t)$ of Eqs (3.1) describes the evolution of the derivatives with respect to the initial data on the manifold $\mathfrak{N}_1$ along the extremal $\hat{x}(\cdot)$, $\hat{\psi}(\cdot)$.

Then, the corresponding solution $W_1(t)$ of Riccati's equation gives the Hessian of Bellman's function $S_1(t, x)$ of the field

$$W_1(t) = \frac{\partial^2 S_1}{\partial x^2}(t, \hat{x}(t)) \tag{3.3}$$

Proof. It follows from formula (2.6) that the vector ψ_1 at any point (t, x), covered by the field of the extremal, specifies the gradient of Bellman's function

$$\psi_1(t, x) = \frac{\partial S_1(t, x)}{\partial x} \tag{3.4}$$

The matrix p is, by definition, the matrix of derivatives of ψ_1 with respect to initial values $_1$, while q is a matrix of the derivatives of x with respect to initial values σ_1. Consequently, the matrix q^{-1} is the derivatives of σ_1 with respect to x, i.e.

$$pq^{-1} = \frac{\partial\psi_1(t,x)\partial\sigma_1}{\partial\sigma_1\ \partial x} = \frac{\partial^2 S_1(t,x)}{\partial x^2} \tag{3.5}$$

Similarly we have for the field $\mathfrak{P}_2$

$$pq^{-1} = -\frac{\partial^2 S_2(t,x)}{\partial x^2} \tag{3.6}$$

The minus sign appears due to the fact that $\psi_2(t,x) = -\frac{\partial S_2(t,x)}{\partial x}$, since, to identify S_2 with J, one must deal with the lower limit of integration.

Consider the field of extremals $\mathfrak{P}_1$ for the left manifold M_1 and the field of extremals $\mathfrak{P}_2$ for the right manifold M_2. Since both fields include the extremal $\hat{x}(\cdot)$, we have

$$\hat{\psi}_1(t) = \hat{\psi}_2(t), \quad t \in [\hat{t}_1, \hat{t}_2] \tag{3.7}$$

Consequently

$$\frac{\partial S_1(t,\hat{x}(t))}{\partial x} = -\frac{\partial S_2(t,\hat{x}(t))}{\partial x} \tag{3.8}$$

i.e. the tangent plane to the level surfaces of the functions S_1 and S_2 at points of the trajectory $x(\cdot)$ coincide and are oppositely orientated.

It is shown in the theorems which follow below that the necessary condition for optimality is the non-negativity, while the sufficient condition is the positive definiteness, of the quadratic form with matrix $(W_1 - W_2)$.

Theorem 2. Suppose Assumptions 1, 2 and 3 for the trajectory $\hat{x}(\cdot)$, which satisfy Pontryagin's maximum principle (2.1)–(2.3), are satisfied.

Then the necessary condition for the trajectory $\hat{x}(\cdot)$ to deliver a weak minimum to functional (1.1), is the non-negativity of the quadratic form with matrix $(W_1(\tau) - W_2(\tau))$ for the $t \in (\hat{t}_0, \hat{t}_1)$.

Proof. For the trajectory $\hat{x}(\cdot)$, which gives a weak minimum, we will assume the opposite, i.e. that there is an instant τ and a vector ξ such that

$$(W_1(\tau)\xi,\xi) - (W_2(\tau)\xi,\xi) < 0 \tag{3.9}$$

We will consider the trajectory $x_1(\cdot)$ of the field $\mathfrak{P}_1$, which ends at the point $(\tau, \hat{x}(\tau) + \xi)$. Such a trajectory exists, since, without loss of generality, by virtue of the uniformity, we can assume that the vector ξ is as small as desired. Exactly the same trajectory $x_2(\cdot)$ of the field $\mathfrak{P}_2$ exists, and begins at the point $(\tau, \hat{x}(\tau) + \xi)$. Note that for any point $(t,x) \in N_1 \cap N_2$ and for the composite extremal $(x(\cdot), u(\cdot))$, made up of the fields $\mathfrak{P}_1$ and $\mathfrak{P}_2$, passing through the point (t,x), we have the equality.

$$S_1(t,x) + S_2(t,x) = J(u(\cdot)) \tag{3.10}$$

Consider the Taylor expansions

$$S_k(\tau,\hat{x}(\tau)+\xi) = S_k(\tau,\hat{x}(\tau)) + \frac{\partial S_k(\tau,\hat{x}(\tau))}{\partial x}\xi + \frac{(-1)^{k-1}}{2}(W_k(\tau)\xi,\xi) + o(|\xi|^2) \tag{3.11}$$

Combining formulae (3.11) for $k = 1$ and $k = 2$ and taking equality (3.8) into account, we obtain

$$(S_1+S_2)(\tau,\hat{x}(\tau)+\xi) - (S_1+S_2)(\tau,\hat{x}(\tau)) = \frac{1}{2}((W_1 - W_2)(\tau)\xi,\xi) + o(|\xi|^2) < 0 \tag{3.12}$$

By virtue of relations (3.9) and (3.10) we attempted to obtain a diminishing of the functional J. In view of the arbitrary smallness of ξ and the continuity of the field $\mathfrak{P}_i$, the angles of inclination of both the left and right halves of the curve constructed at the point where they join differ as little as desired from the angle of inclination of the extremal $\hat{x}(\tau)$. Consequently, we can smooth the extremal $x(\cdot)$ at the corner point, without violating inequality (3.12) and without departing the C^1-neighbourhood of the trajectory $\hat{x}(\cdot)$. Consequently, a weak minimum is not reached on the trajectory $\hat{x}(\cdot)$, which contradicts the assumption made above.

Theorem 3. Suppose Assumptions 1, 2 and 3 for the trajectory $\hat{x}(\cdot)$, satisfying Pontryagin's maximum principle (2.1)–(2.3), are satisfied.

Then the sufficient condition for the trajectory $\hat{x}(\cdot)$ to deliver a strong minimum to the functional (1.1) is that the quadratic form with matrix $(W_1(\tau) - W_2(\tau))$ to be positive definite for a certain $\tau \in (\hat{t}_1, \hat{t}_2)$.

Proof. Consider the arbitrary permissible trajectory $x(\cdot) \subset N_1 \cap N_2$, corresponding to the control $u(\cdot)$ and lying in the ε-neighbourhood (in the C topology) of the curve $\hat{x}(\cdot)$. Then $|x(\tau) - \hat{x}(\tau)| < \varepsilon$, and consequently $x(\tau) = \hat{x}(\tau) + \xi$. Suppose the trajectory $x(\cdot)$ intersects the manifold M_1 at $t = t_1$ and M_2 at $t = t_2$. Note that $\tau \in (t_1, t_2)$ for sufficiently small ε. In view of the definition of the function $S_k(t, x)$ we have the inequalities

$$\int_{t_1}^{\tau} f(t, x(t), u(t))dt \geq S_1(\tau, x(\tau)), \quad \int_{\tau}^{t_2} f(t, x(t), u(t))dt \geq S_2(\tau, x(\tau))$$

Adding, we obtain

$$J(u(\cdot)) \geq S_1(\tau, x(\tau)) + S_2(\tau, x(\tau))$$

Again using Taylor expansion (3.12), we obtain

$$J(u(\cdot)) - J(\hat{u}(\cdot)) \geq \frac{1}{2}((W_1 - W_2)(\tau)\xi, \xi) + o(|\xi^2|) > 0$$

Remark 1. Problems in which the boundary values for the left and right ends of the trajectory are encountered in a single common formula, can be reduced to a problem with separated conditions for the ends, considered in this paper, by introducing auxiliary variables.

Consider, for example, the problem of minimizing the functional

$$J(u(\cdot)) = \int_0^T f(t, x, u)dt$$

with the constraints

$$\dot{x} = \varphi(t, x, u), \quad \Phi(x(0), x(T)) = 0; \quad u(t) \in U$$

We will introduce the new variables

$$y(t) = x(t), \quad z(t) = x(T - t)$$

Then the boundary conditions take the form

$$\Phi(y(0), z(0)) = 0, \quad \Phi(z(T), y(T)) = 0$$

2. It follows from Theorem 2 and 3 that if the matrix $(W_1 - W_2)$ is positive for one value of t, it will remain non-negative for all $t \in (\hat{t}_1, \hat{t}_2)$.

3. A check of the necessary and sufficient conditions, presented in Theorem 2 and 3, gives rise to no additional difficulties compared with a check of Assumption 3. No additional integration of Riccati's matrix equation (3.2) is required, since the solution of this equation is expressed in terms of the solutions of the Euler–Jacobi equations which have already been solved when checking Assumption 3. Only the question of whether the matrices obtained are positive definite remains.

The results obtained also hold in the more general situation, in particular, when there are switching surfaces. To overcome the difficulties that arise in this case one can use the technique of piecewise-continuous Lagrange manifolds, developed previously in [2]. Moreover, since the solutions of the equations in variations, when the switching surfaces intersect, undergo jumps, it is necessary to consider discontinuous solutions of Riccati's equations [3].

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